Available online at www.sciencedirect.com

SCIENCE DIRECT®

Theoretical Computer Science 346 (2005) 335–357

Theoretical Computer Science

www.elsevier.com/locate/tcs

A sufficient condition for non-uniqueness in binary tomography with absorption

Attila Kuba[a,*], Murice Nivat[b]

[a] *Department of Image Processing and Computer Graphics, University of Szeged, Árpád tér 2, H-6720 Szeged, Hungary*

[b] *Laboratoire d'Informatique Algorithmique: Fondements et Applications, Université Paris 7 Denis-Diderot, Paris, France*

Abstract

A new kind of discrete tomography problem is introduced: the reconstruction of discrete sets from their absorbed projections. A special case of this problem is discussed, namely, the uniqueness of the binary matrices with respect to their absorbed row and column sums when the absorption coefficient is $\mu = \log((1+\sqrt{5})/2)$. It is proved that if a binary matrix contains a special structure of 0s and 1s, called alternatively corner-connected component, then this binary matrix is non-unique with respect to its absorbed row and column sums. Since it has been proved in another paper [A. Kuba, M. Nivat, Reconstruction of discrete sets with absorption, Linear Algebra Appl. 339 (2001) 171–194] that this condition is also necessary, the existence of alternatively corner-connected component in a binary matrix gives a characterization of the non-uniqueness in this case of absorbed projections.

Keywords: Discrete tomography; Binary matrices; Absorbed projection

1. Introduction

Discrete tomography (DT) deals with the problem of reconstructing functions with given discrete ranges from weighted sums/integrals over subsets/subspaces (e.g., straight lines

* Corresponding author.
E-mail addresses: kuba@inf.u-szeged.hu (A. Kuba), tcsmn@club-internet.fr (M. Nivat).

0304-3975/$ - see front matter
doi:10.1016/j.tcs.2005.08.024

or planes) of their domain. It has applications, for example, in electron microscopy [7,13] and medicine [9]. The book [6] provides an overview of the foundations, algorithms, and applications of discrete tomography.

In this paper we consider a generalisation of the DT problem. Let us suppose that the function to be reconstructed represents a discrete set, which is in some known absorbing material and, accordingly, the measurements are the absorbed sums taken on straight lines. It can be considered as the basic model of the *emission discrete tomography*, or EDT, where the elements of the discrete set are discrete sources emitting unit energy and the measurements represent the partially absorbed energy detected along straight lines. (In this sense the classical model of DT can be considered as the special case of the EDT when there is no absorption.) In the last 2 years a few other papers have been published related to EDT, see, for example [2,3].

First, we pose the reconstruction problem of discrete sets with absorption in Section 2. Then we select a mathematically interesting special absorption value $\mu = \log((1+\sqrt{5})/2)$ and investigate the question of uniqueness when the discrete set can be represented by a binary matrix and the absorbed row and column sums of this matrix are given. In Section 3 we show that in this case the absorbed row and column sums can be considered as finite β_0-representations (a terminology used in numeration systems), where $\beta_0 = e^{\mu}$. Section 4 deals with those transformations, called switchings, when the 0s and 1s of a certain subset (called switching pattern) are switched to each other but the absorbed projections of the subset remain the same. Clearly, if a binary matrix contains a switching pattern then it is non-unique, because we can get another binary matrix with the same absorbed projections by switching transformation. We determine the switching patterns in both one-dimensional (1D) and two-dimensional (2D) cases. It is also shown how more complex switching patterns, like alternatively corner-connected components, can be created from simple, elementary switching patterns.

In another paper [10] we have proved that the existence of alternatively corner-connected components in a binary matrix is necessary for the non-uniqueness, therefore it is necessary and sufficient for the non-uniqueness of binary matrices with respect to their absorbed row and column sums.

2. Absorption and reconstruction of discrete sets

2.1. Absorption

Consider a ray (e.g, light or X-ray) passing through a homogeneous material. It is well-known that a part of the ray will be absorbed in the material. Quantitatively, let I_0 and I denote the initial and the detected intensities (number of photons/s) of the ray. Then

$$I = I_0 e^{-\mu x}, \tag{1}$$

where $\mu \geqslant 0$ denotes the absorption coefficient of the material and x is the length of the path of the ray in the material (see Fig. 1).

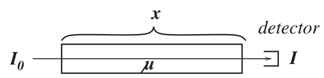


Fig. 1. A part of the initial intensity I_0 of the ray is absorbed in the homogeneous material having absorption coefficient μ.

2.2. The reconstruction problem for discrete sets

Let $\mathbb{Z}^d$ denote the d-dimensional integer lattice ($d \geqslant 2$). The non-zero vectors of $\mathbb{Z}^d$ are the *lattice directions*. The *lattice lines* are the lines of the d-dimensional Euclidean space which are parallel to a lattice direction and pass through at least one lattice point in $\mathbb{Z}^d$.

The finite subsets of $\mathbb{Z}^d$ are called *discrete sets*. Let F be a discrete set. The *projection of F along a lattice line ℓ* is defined as

$$[\mathcal{P}F](\ell) = |F \cap \ell|,$$

where $|.|$ denotes the cardinality of the argument set.

Let $\mathcal{E}$ be a class of discrete sets and $\mathcal{L}$ be a finite collection of lattice lines. Then the *reconstruction problem* for $\mathcal{E}$ and $\mathcal{L}$ can be posed as

RECONSTRUCTION $D(\mathcal{E}, \mathcal{L})$.

Given: *Function $p : \mathcal{L} \longrightarrow \mathbb{N}_0$ ($\mathbb{N}_0$ denotes the set of nonnegative integers).*

Task: *Construct a discrete set $F \in \mathcal{E}$ such that*

$$[\mathcal{P}F](\ell) = p(\ell)$$

for all $\ell \in \mathcal{L}$.

Many results connected to this reconstruction problem have been published in the last years. A summary of these results are in [6]. Consider, for example, the problem of uniqueness in the case of (2D) binary matrices.

UNIQUENESS $D2D(A)$.

Given: *$m, n \in \mathbb{N}$, and a binary matrix A with size $m \times n$.*

Question: *Does there exist a different binary matrix A' with the same size such that the row and column sums of A and A' are the same?*

It has been shown [12] that a binary matrix is non-unique with respect to its row and column sums if and only if it has a sub-matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

called *switching component*.

A new kind of DT problem, the reconstruction of discrete sets from their absorbed projections, was introduced in [10]. A special case of this problem was discussed, namely, the

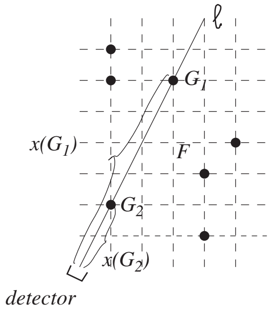


Fig. 2. Computation of the projection of the discrete set F along the lattice line ℓ. The elements of F are denoted by bold points. Here $F \cap \ell = \{G_1, G_2\}$.

uniqueness of 2D binary matrices with respect to their absorbed row and column sums when the absorption is represented by the constant β_0 of (7).

2.3. The reconstruction problem for discrete sets with absorption

Consider now the corresponding reconstruction problem in the case of absorption. For the sake of simplicity we suppose that the whole d-dimensional Euclidean space is uniformly filled with the material having absorption coefficient $\mu \geqslant 0$. Then the *projection with absorption μ of a discrete set F along a lattice line ℓ* is defined according to (1) by

$$[\mathcal{P}_\mu F](\ell) = \sum_{G \in F \cap \ell} \mathrm{e}^{-\mu \cdot x(G)}, \tag{2}$$

where $x(G)$ denotes the distance between the point G and the detector placed on the lattice line ℓ (see Fig. 2).

The *reconstruction problem with absorption μ* for a class of discrete sets, $\mathcal{E}$, knowing the projections along all directed lines of $\mathcal{L}$ (we have to have a detector location for each line ℓ) can be posed as

RECONSTRUCTION $DA(\mu, \mathcal{E}, \mathcal{L})$.

Given: *Function $p : \mathcal{L} \longrightarrow \mathbb{R}_0$ ($\mathbb{R}_0$ denotes the set of nonnegative real numbers).*

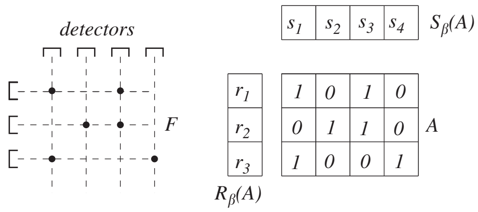
Task: *Construct a discrete set $F \in \mathcal{E}$ such that*

$$[\mathcal{P}_\mu F](\ell) = p(\ell)$$

for all $\ell \in \mathcal{L}$.

2.4. Reconstruction of 2D discrete sets with absorption

Consider now the 2D integer lattice $\mathbb{Z}^2$. Let m and n be positive integers. A *discrete rectangle* with size $m \times n$ is a special discrete set of $\mathbb{Z}^2$ determined as the intersection of m consecutive horizontal lattice lines with n consecutive vertical lattice lines. If F is a



(a) (b)

Fig. 3. A 2D discrete set F, the corresponding binary matrix A, and their horizontal and vertical projections. (a) The set F and the detectors measuring its horizontal and vertical projections. (b) The binary matrix A and its absorbed row and column sums.

discrete set in $\mathbb{Z}^2$, then there is an $m \times n$ discrete rectangle containing F, it is called *containing* (*discrete*) *rectangle*. (For the sake of simplicity we can take the smallest containing rectangle in the following.)

Let us suppose that the horizontal and vertical projections of F are measured by detectors placed in the next column to left and in the next row to above, respectively, of the containing rectangle (see Fig. 3(a)). Then the absorbed projections can be computed according to (2). For example, in the case of the discrete set F given in Fig. 3(a), the absorbed projections along the horizontal lattice lines of the containing rectangle are $r_1 = \mathrm{e}^{-\mu\cdot 1} + \mathrm{e}^{-\mu\cdot 3}$, $r_2 = \mathrm{e}^{-\mu\cdot 2} + \mathrm{e}^{-\mu\cdot 3}$, and $r_3 = \mathrm{e}^{-\mu\cdot 1} + \mathrm{e}^{-\mu\cdot 4}$.

Now, we introduce an equivalent representation of the 2D discrete sets and their absorbed horizontal and vertical projections. The containing rectangle including F can be represented by a binary matrix $A = (a_{ij})_{m\times n}$ as follows: $a_{ij} = 1$ if the lattice point corresponding to (i, j) is an element of F, $a_{ij} = 0$ otherwise. In order to use the generally accepted notation of numeration systems [11], let us introduce

$$\beta = \mathrm{e}^{\mu}. \tag{3}$$

Clearly, $\beta \geqslant 1$. Then we can define the *absorbed row* and *column sums* of A, $R_\beta(A)$ and $S_\beta(A)$, respectively, as

$$R_\beta(A) = (r_1, \ldots, r_m), \tag{4}$$

where

$$r_i = \sum_{j=1}^{n} a_{ij}\beta^{-j}, \quad i = 1, \ldots, m$$

and

$$S_\beta(A) = (s_1, \ldots, s_n), \tag{5}$$

where

$$s_j = \sum_{i=1}^{m} a_{ij}\beta^{-i}, \quad j = 1, \ldots, n.$$

For example, in the case of matrix A given in Fig. 3(b) $R_\beta(A) = (\beta^{-1} + \beta^{-3},\ \beta^{-2} + \beta^{-3},\ \beta^{-1} + \beta^{-4})$ and $S_\beta(A) = (\beta^{-1} + \beta^{-3},\ \beta^{-2}, \beta^{-1} + \beta^{-2},\ \beta^{-3})$.

We say that two binary matrices with the same sizes are *tomographically equivalent* if they have the same absorbed row and column sums.

Then the *uniqueness problem of 2D discrete sets* (or, equivalently, *of binary matrices*) *with absorption* knowing the absorbed projections along horizontal and vertical lines can be posed as

UNIQUENESS $DA2D(\beta, A)$.

Given: $\beta \geqslant 1$, *m, n, and a binary matrix A with size $m \times n$.*

Question: *Does there exist a different binary matrix A' with the same size such that A and A' are tomographically equivalent with respect to their absorbed projections R_β and S_β?*

If $\beta = 1$ then we have the classical uniqueness problem of binary matrices without absorption (see e.g. [4,8]).

Binary matrices with absorption can be reconstructed even only from their row sums in certain cases. For example, if β and n have the following property: for any positive integers t, z, and $1 \leqslant p_1 < \cdots < p_t \leqslant n$, $1 \leqslant q_1 < \cdots < q_z \leqslant n$

$$\beta^{-p_1} + \cdots + \beta^{-p_t} = \beta^{-q_1} + \cdots + \beta^{-q_z}$$

implies that

$$t = z \quad \text{and} \quad p_1 = q_1, \ldots, p_t = q_t.$$

In this case each row of A is uniquely determined by its absorbed row sum, and so A is uniquely determined by $R_\beta(A)$. For example, if $\beta \geqslant 2$ then we have this property for any $n \geqslant 1$.

But how can we do reconstruction if β and n do not have this property? Select, for example, the case $\beta = \beta_0$, where

$$\beta_0^{-1} = \beta_0^{-2} + \beta_0^{-3} \tag{6}$$

giving

$$\beta_0 = \frac{1 + \sqrt{5}}{2}. \tag{7}$$

(The other solutions of (6), namely $(1 - \sqrt{5})/2$ and 0, are not applicable in this case; c.f., (3).) In this case $\mu = \log((1 + \sqrt{5})/2)$ which gives a mathematically interesting case that will be analysed in detail in this paper.

3. β_0-representation

Consider the absorbed row and column sums of the binary matrix A in the case of $\beta = \beta_0$:

$$r_i = \sum_{j=1}^{n} a_{ij} \beta_0^{-j}, \quad i = 1, \ldots, m \tag{8}$$

and

$$s_j = \sum_{i=1}^{m} a_{ij} \beta_0^{-i}, \quad j = 1, \ldots, n. \tag{9}$$

Using the terminology of numeration systems [11] we can say that the finite (binary) word $a_{i1} \cdots a_{in}$ is a (*finite*) *representation in base* β_0 (or a *finite* β_0-*representation*) of r_i for each $i = 1, \ldots, m$, and, similarly, $a_{1j} \cdots a_{nj}$ is a β_0-representation of s_j for each $j = 1, \ldots, n$. The Eqs. (8) and (9) mean also that the absorbed row and column sums of A are nonnegative real numbers having a finite β_0-representation with n and m binary digits, respectively (including the eventually ending zeros).

Since we deal only with finite length β_0-representations in this paper, the β_0-representation always means finite length β_0-representation.

Let B_k denote the set of nonnegative real numbers having a β_0-representation with k binary digits ($k > 1$), formally,

$$B_k = \left\{ \sum_{i=1}^{k} a_i \beta_0^{-i} \mid a_i \in \{0, 1\} \right\}.$$

Then

$$r_i \in B_n, \quad i = 1, \ldots, m$$

and

$$s_j \in B_m, \quad j = 1, \ldots, n,$$

are necessary conditions for the existence of a matrix A with

$$R_{\beta_0}(A) = (r_1, \ldots, r_m) \quad \text{and} \quad S_{\beta_0}(A) = (s_1, \ldots, s_n).$$

4. Switchings

Switching is, roughly, a transformation of β_0-representations by which certain 0s and 1s are replaced to each other such that the represented values remain the same. As it will be proven in this section, switchings play a basic role in the uniqueness problem $DA2D(\beta_0, A)$.

4.1. 1D switchings

The β_0-representation is generally nonunique, because there are binary words with the same length representing the same number. For example, on the base of (6), it is easy to check the following equality between the 3-digit-length β_0-representations of $1/\beta_0$

$$100 = 011 \tag{10}$$

which is the most simple example of switching.

Furthermore (6) may allow us to define more general switchings: Let $a_1 \cdots a_k$ be a k-digit-length β_0-representation and let I be a subset of $\{1, \dots, k\}$. We say that I is a set of *switching positions* of $a_1 \cdots a_k$ if by replacing (switching) the 1s and 0s to each other in the positions of I the new (switched) word represents the same number as $a_1 \cdots a_k$. This transformation of the β_0-representations is called *1D switching*. As an example, let us take Eq. (10). Here all positions, 1, 2, and 3, are switching positions of 100. Another example can be shown by using 100 00. Since

$$100\,00 = 010\,11,$$

$I = \{1, 2, 4, 5\}$ is a set of switching positions of 100 00 (and of 010 11).

If there is one of the sub-words 011 and 100 in a β_0-representation then it can be replaced by the other one without changing the value of the representation, i.e., it is a special kind of switching. It is called *1D elementary switching*. For example, a 1D elementary switching can be done in the positions 2, 3, and 4 of the word 010 00 getting the word 001 10 representing the same number. The words 011 and 100 are called 0-*type* and 1-*type* 1*D elementary switching words*, respectively, also the *switching pair* expression can be used.

As direct consequences of (10), it is easy to see that

$$\begin{aligned} 011 &= 100, \\ 01x_3 11 &= 10x_3 00, \\ 01x_3 1x_5 11 &= 10x_3 0x_5 00, \\ 01x_3 1x_5 1x_7 11 &= 10x_3 0x_5 0x_7 00, \dots \end{aligned} \tag{11}$$

where $x_3, x_5, x_7, \dots$ denotes the positions where both β_0-representations have the same (but otherwise arbitrary) binary digit. For example, the second equality of (11) can be proved as follows:

$$\begin{aligned} \text{if} \quad & x_3 = 0, \\ \text{then} \quad & 01x_3 11 = 010\,11 = 011\,00 = 100\,00 = 10x_3 00, \\ \\ \text{if} \quad & x_3 = 1, \\ \text{then} \quad & 01x_3 11 = 011\,11 = 100\,11 = 101\,00 = 10x_3 00, \end{aligned} \tag{12}$$

that is, we got the necessary representation after two consecutive 1D elementary switchings in both cases. The third, fourth, etc. equalities of (11) can be proved similarly by applying three, four, etc. 1D elementary switchings.

Accordingly, if in a β_0-representation there is one of the words listed in either side of the Eq. (11) then it can be replaced (switched) by the word on the other side of the corresponding Eq. (11) without changing the value of the representation. This transformation of binary words is called *1D composite switching*. The words starting with 0 (resp. 1) in the Eq. (11) will be called *0-type* (resp., *1-type*) *1D composite switching words*, respectively.

For example, 101 00 is a 1-type composite switching word being on the left side of the second equation of (11).

Now we are going to show that the 1D composite switchings can be composed from 1D elementary switchings as follows. Let $a_1 \cdots a_k$ and $b_1 \cdots b_l$ (k and l are positive odd integers) be two 1D switching words of the same type (i.e., either $a_1 = b_1 = 0$ or $a_1 = b_1 = 1$). Their composition, denoted by $*$, is defined as

$$a_1 \cdots a_k * b_1 \cdots b_l = a_1 \cdots a_{k-1}\_\_b_2 \cdots b_l,$$

where __ indicates that the kth position of the resulting composition word is undefined. For example,

$$\begin{aligned} 011 * 011 &= 01\_\_11, \\ 100 * 100 &= 10\_\_00, \\ 01x_3 11 * 011 &= 01x_3 1\_\_11, \\ 10x_3 0x_5 0x_7 00 * 10y_3 0y_5 00 &= 10x_3 0x_5 0x_7 0\_\_0y_3 0y_5 00, \end{aligned}$$

where x_3, x_5, x_7, and y_3, y_5 denotes the positions where both β_0-representations have the same (but otherwise arbitrary) binary digit.

Now, we are going to prove that any finite β_0-representation of a number can be got from its any other β_0-representation by 1D elementary switchings.

Lemma 1. *Let $a_1 \cdots a_k$ and $b_1 \cdots b_k$ be different, k-digit-length β_0-representations of the same number. Then $b_1 \cdots b_k$ can be get from $a_1 \cdots a_k$ by a finite number of switchings.*

Proof. We are going to give a procedure by which suitable switching sub-words can be found in $a_1 \cdots a_k$ and, by switching them, the number of different positions between $a_1 \cdots a_k$ and $b_1 \cdots b_k$ can be decreased until there is no different position.

Let i be the first position ($1 \leqslant i \leqslant k$) where the two representations are different, that is,

$$\begin{aligned} a_1 \cdots a_n &= x_1 \cdots x_{i-1} a_i \cdots a_k \\ &= x_1 \cdots x_{i-1} b_i \cdots b_k = b_1 \cdots b_k, \end{aligned} \tag{13}$$

where $x_1, \ldots, x_{i-1}$ denote positions where the two representations have the same binary digit. Furthermore, let us suppose that $a_i = 1$ and $b_i = 0$, that is,

$$\begin{aligned} a_i \cdots a_k &= 1a_{i+1} \cdots a_k, \\ b_i \cdots b_k &= 0b_{i+1} \cdots b_k. \end{aligned}$$

Note that $k \neq i$ (for otherwise it would follow from (13) that $1 = 0$) and that $k \neq i + 1$ (for otherwise it would follow from (13) that β_0 is an integer). Hence $k > i + 1$. Then $a_{i+1} = 0$ and $b_{i+1} = 1$, as it can be seen indirectly in the following. Otherwise there are two cases:

Case 1: If $a_{i+1} = 1$ then

$$a_i \cdots a_k = 11a_{i+2} \cdots a_k \geqslant 110 \cdots 0. \tag{14}$$

Let us write as many 0's to the end of the binary word on the right side of (14) (if it is not so yet) that the length of the padded binary word is an even number being not less than 4.

Then

$$1100\cdots 00 = 1010\cdots 0\underline{1}1 \\ \qquad > 1010\cdots 0\underline{0}1 = 011\cdots 11, \tag{15}$$

where we have equalities by switchings and strict inequality by changing the 1 to 0 in position last but one (denoted by underlined digit). Since

$$011\cdots 11 \geqslant b_i\cdots b_k \tag{16}$$

from (14)–(16) we got the contradiction $a_i\cdots a_k > b_i\cdots b_k$.

Case 2: If $b_{i+1} = 0$ then let us follow a similar idea as in Case 1:

$$a_i\cdots a_k = 1a_{i+1}\cdots a_k \geqslant 100\cdots 0. \tag{17}$$

Let us write as many 0's to the end of the binary word on the right side of (17) (if it is not so yet) that the length of the padded binary word is an even number being not less than 4. Then

$$1000\cdots 00 = 0101\cdots 01\underline{1}0 \\ \qquad > 0101\cdots 01\underline{0}0 = 0011\cdots 11, \tag{18}$$

where we have equalities by switchings and strict inequality by changing the 1 to 0 in position last but one (denoted by underlined digits). Since

$$0011\cdots 11 \geqslant 00b_{i+2}\cdots b_k = b_i\cdots b_k \tag{19}$$

from (17)–(19) we got again the contradiction $a_i\cdots a_k > b_i\cdots b_k$.

Therefore, $a_{i+1} = 0$ and $b_{i+1} = 1$. That is, if $a_i\cdots a_k$ and $b_i\cdots b_k$ represent the same number then

$$a_i\cdots a_k = 10a_{i+2}\cdots a_k, \\ b_i\cdots b_k = 01b_{i+2}\cdots b_k$$

necessarily.

Consider now the next position, $i+2$, in $a_i\cdots a_k$ and $b_i\cdots b_k$. Then there are three cases to be studied.

Case 1: If $a_{i+2} = 0$ and $b_{i+2} = 1$ then we have the switching sub-words $a_ia_{i+1}a_{i+2} = 100$ and $b_ib_{i+1}b_{i+2} = 011$. Therefore $a_ia_{i+1}a_{i+2}$ can be switched to $b_ib_{i+1}b_{i+2}$ getting a new representation $a_1\cdots a_{i-1}\ b_ib_{i+1}b_{i+2}a_{i+3}\cdots a_k$ which can be different from $b_1\cdots b_k$ only from the position $i+3$. If $a_1\cdots a_{i-1}b_ib_{i+1}\ b_{i+2}a_{i+3}\cdots a_k$ and $b_1\cdots b_k$ are different then we can start the same procedure to find a switching in $a_{i+3}\cdots a_k$ and $b_{i+3}\cdots b_k$, but now we have shorter representations to prove the lemma. Otherwise the two representations are the same, so we got $b_1\cdots b_k$ from $a_1\cdots a_k$ by one switching which means that the lemma is satisfied.

Case 2: If $a_{i+2} = 1$ and $b_{i+2} = 0$ then

$$a_i\cdots a_k = 101a_{i+3}\cdots a_k \geqslant 101\,00\cdots 0. \tag{20}$$

Let us write as many 0's to the end of the binary word on the right side of (20) (if it is not so yet) that the length of the padded binary word is an even number being not less than 6. Then

$$\begin{aligned} 101\,00\cdots 00 &= 011\,10\cdots 01\underline{1}0 > 011\,10\cdots 01\underline{0}0 \\ &= 01\underline{10}11\cdots 111 > 01\underline{01}11\cdots 11, \end{aligned} \tag{21}$$

where we have equalities by switchings and strict inequalities by changing the corresponding underlined digits in both sides. Since

$$010\,11\cdots 11 \geqslant 010 b_{i+3}\cdots b_k = b_i\cdots b_k \tag{22}$$

from (20)–(22) we got the contradiction $a_i\cdots a_k > b_i\cdots b_k$. That is, this case is impossible.

Case 3: The last case is when $a_{i+2} = b_{i+2}$, that is $a_{i+2} = b_{i+2} = 0$ or $a_{i+2} = b_{i+2} = 1$. Let us denote this common binary digit by x_{i+2}. In this case we have to continue the search for a composite switching word.

That is, let us continue the procedure with the sub-words

$$\begin{aligned} a_i\cdots a_k &= 10x_{i+2}a_{i+3}\cdots a_k, \\ b_i\cdots b_k &= 01x_{i+2}b_{i+3}\cdots b_k, \end{aligned}$$

where x_{i+2} denotes the position where the two representations have the same binary digit.

Now, we are going to show that $a_{i+3} = 0$ and $b_{i+3} = 1$ (note that $k > i+2$, for otherwise it would follow that $10x_{i+2} = 01x_{i+2}$). Otherwise, there are two cases.

Case 1: If $a_{i+3} = 1$ then

$$a_i\cdots a_k = 10x_{i+2}1a_{i+4}\cdots a_k \geqslant 10x_{i+2}10\cdots 0. \tag{23}$$

Let us write as many 0's to the end of the binary word on the right side of (23) (if it is not so yet) that the length of the padded binary word is an even number being not less than 6. Then

$$\begin{aligned} &10x_{i+2}100\cdots 000 = 10x_{i+2}010\cdots 0\underline{1}1 \\ &> 10x_{i+2}010\cdots 0\underline{0}1 = 10x_{i+2}0011\cdots 11 \\ &= 01x_{i+2}111\cdots 11, \end{aligned} \tag{24}$$

where we have equalities by switchings and strict inequality by changing the 1 to 0 in position indicated by underlined digit. Since

$$01x_{i+2}11\cdots 11 \geqslant 01x_{i+2}b_{i+3}\cdots b_k = b_i\cdots b_k \tag{25}$$

from (23)–(25) we got the contradiction $a_i\cdots a_k > b_i\cdots b_k$. That is, this case is impossible.

Case 2: If $b_{i+3} = 0$ then let us follow a similar idea as in Case 1:

$$a_i\cdots a_k = 10x_{i+2}a_{i+3}\cdots a_k \geqslant 10x_{i+2}0\cdots 0. \tag{26}$$

Let us write as many 0's to the end of the binary word on the right side of (26) (if it is not so yet) that the length of the padded binary word is an even number being not less than 7.

Then

$$\begin{aligned}
&10x_{i+2}0000\cdots000 = 01x_{i+2}1100\cdots000\\
&= 01x_{i+2}1010\cdots0\underline{1}1 > 01x_{i+2}1010\cdots0\underline{0}1\\
&= 01x_{i+2}0011\cdots111,
\end{aligned}\tag{27}$$

where we have equalities by switchings and strict inequality by changing the 1 to 0 (in position indicated by underlined digit). Since

$$01x_{i+2}011\cdots11 = 01x_{i+2}0b_{i+4}\cdots b_k \geqslant b_i\cdots b_k \tag{28}$$

from (26)–(28) we got the contradiction $a_i\cdots a_k > b_i\cdots b_k$. That is, this case is also impossible.

Therefore, $a_{i+3} = 0$ and $b_{i+3} = 1$.

In the following we have to study the possibilities when

$$\begin{aligned}
a_i\cdots a_k &= 10x_{i+2}0a_{i+4}\cdots a_k,\\
b_i\cdots b_k &= 01x_{i+2}1b_{i+4}\cdots b_k.
\end{aligned}$$

Consider now the next position, $i+4$, in $a_i\cdots a_k$ and $b_i\cdots b_k$ (note that $k > i+3$, for otherwise it would follow that $10x_{i+2}0 = 01x_{i+2}1$). Then there are three cases to be studied.

Case 1: If $a_{i+4} = 0$ and $b_{i+4} = 1$ then we have the switching sub-words $a_ia_{i+1}a_{i+2}a_{i+3}a_{i+4} = 10x_{i+2}00$ and $b_ib_{i+1}b_{i+2}b_{i+3}b_{i+4} = 01x_{i+2}11$ (see (11)) and we can switch $a_ia_{i+1}a_{i+2}a_{i+3}a_{i+4}$ and $b_ib_{i+1}b_{i+2}b_{i+3}b_{i+4}$ getting a new representation, $a_1\cdots a_{i-1}b_ib_{i+1}\,b_{i+2}b_{i+3}b_{i+4}a_{i+5}\cdots a_k$, which can be different from $b_1\cdots b_k$ only from the position $i+5$. If the new representation is different from $b_1\cdots b_k$ then we can start the same procedure to find a switching in $a_{i+5}\cdots a_k$ and $b_{i+5}\cdots b_k$, but now we have shorter representations to prove the lemma. Otherwise, the two representations are the same, so we got $b_1\cdots b_k$ from $a_1\cdots a_k$ by one switching which means that the lemma is satisfied.

Case 2: If $a_{i+4} = 1$ and $b_{i+4} = 0$ then we get a contradiction in the same way as in the case of $a_{i+2} = 1$ and $b_{i+2} = 0$. That is, this case is impossible.

Case 3: The last case is when $a_{i+4} = b_{i+4}$. In this case we have to continue the search for a composite switching word.

That is, we have to continue the procedure according to Case 3 with the sub-words

$$\begin{aligned}
a_i\cdots a_k &= 10x_{i+2}0x_{i+4}a_{i+5}\cdots a_k,\\
b_i\cdots b_k &= 01x_{i+2}1x_{i+4}b_{i+5}\cdots b_k,
\end{aligned}$$

where x_{i+2} and x_{i+4} are in the positions where the two representations have the same binary digits.

Then we can show that $a_{i+5} = 0$ and $b_{i+5} = 1$ as it was in the case of $a_{i+3} = 0$ and $b_{i+3} = 1$. That is,

$$\begin{aligned}
a_i\cdots a_k &= 10x_{i+2}0x_{i+4}0a_{i+6}\cdots a_k,\\
b_i\cdots b_k &= 01x_{i+2}1x_{i+4}1b_{i+6}\cdots b_k.
\end{aligned}$$

And so on. Generally, $a_{i+2l} = b_{i+2l}(= x_{i+2l})$, $a_{i+2l+1} = 0$, and $b_{i+2l+1} = 1$ for $l = 1, 2, \ldots$. But this procedure is finished after a finite number of steps, therefore, there is an index value l such that $a_{i+2l+1} = a_{i+2l+2} = 0$ and $b_{i+2l+1} = b_{i+2l+2} = 1$:

$$a_i \cdots a_k = 10x_{i+2}0x_{i+4}0 \cdots 0x_{i+2l}00a_{i+2l+3} \cdots a_k,$$
$$b_i \cdots b_k = 01x_{i+2}1x_{i+4}1 \cdots 1x_{i+2l}11b_{i+2l+3} \cdots b_k.$$

Then we have the switching sub-words $10x_{i+2}0x_{i+4}0 \cdots 0x_{i+2l}00$ and $01x_{i+2}1x_{i+4}1x_{i+6}$ $1 \cdots 1x_{i+2l}11$ (see (11)) and we can switch $a_i \cdots a_{i+2l+2}$ and $b_i \cdots b_{i+2l+2}$ getting a new representation, $a_1 \cdots a_{i-1}b_ib_{i+1} \cdots b_{i+2l+2}a_{i+2l+3} \cdots a_k$ which can be different from $b_1 \cdots b_k$ only from the position $i + 2l + 3$. If the new representation is different from $b_1 \cdots b_k$ then we can start the same procedure to find a switching in $a_{i+2l+3} \cdots a_k$ and $b_{i+2l+3} \cdots b_k$, but now we have shorter representations to prove the lemma. Otherwise, the two representations are the same after this switching.

As a result of this procedure we can say that since the number of different positions of $a_1 \cdots a_k$ and $b_1 \cdots b_k$ decreases by each switchings, we get $b_1 \cdots b_k$ from $a_1 \cdots a_k$ after a finite number of switchings. $\square$

Consequences

(i) If $a_1 \cdots a_k$ and $b_1 \cdots b_k$ are different, k-digit-length β_0-representations of the same number, then there are positions $i, \ldots, i + 2l + 2$ ($l \geqslant 0$, $1 \leqslant i$, $i + 2l + 2 \leqslant k - 2$) such that there is a switching between $a_1 \cdots a_k$ and $b_1 \cdots b_k$ on these positions.

(ii) Let $a_1, \ldots, a_k \in \{0, 1\}$ ($k \geqslant 3$). $a_1 \cdots a_k$ is the only k-digit-length β_0-representation of a real number if and only if it has no switching, or equivalently, it has no sub-word 100 or 011.

4.2. 2D switchings

In this subsection we determine the 2D switchings, i.e., the transformations of binary matrices by which some of their 0s and 1s can be switched to each other such that the absorbed row and column sums remain the same.

4.2.1. Connectedness

Consider now the class of $m \times n$ ($m, n \geqslant 3$) binary matrices with given row and column sum vectors in the case of absorption β_0. Let $Q_{(i,j)}$ ($1 < i < m$, $1 < j < n$) denote the 3×3 discrete square

$$Q_{(i,j)} = \{i-1, i, i+1\} \times \{j-1, j, j+1\}.$$

Let Σ be a set of 3×3 discrete squares of $\{1, \ldots, m\} \times \{1, \ldots, n\}$ and let $Q_{(i,j)}, Q_{(i',j')} \in \Sigma$. Two kinds of connectedness will be defined on 3×3 discrete squares: side-connectedness and corner-connectedness. There is a *side-connection* between $Q_{(i,j)}$ and $Q_{(i',j')}$ if $(i', j') \in \{(i-2, j), (i, j-2), (i, j+2), (i+2, j)\}$. The four squares being side-connected to $Q_{(i,j)}$ are called the *side-neighbours* of $Q_{(i,j)}$. As an illustration see Fig. 4(a). $Q_{(i,j)}$ and $Q_{(i',j')}$ are *corner-connected* if $(i', j') \in \{(i-2, j-2), (i-2, j+2), (i+2, j-2), (i+2, j+2)\}$. The four squares being corner-connected to $Q_{(i,j)}$ are called the *corner-neighbours* of $Q_{(i,j)}$ (see Fig. 4(b)).

There is a *side-chain* between $Q_{(i,j)}$ and $Q_{(i',j')}$ in Σ if a sequence of elements of Σ can be selected such that the first element is $Q_{(i,j)}$ and the last element is $Q_{(i',j')}$ and any two

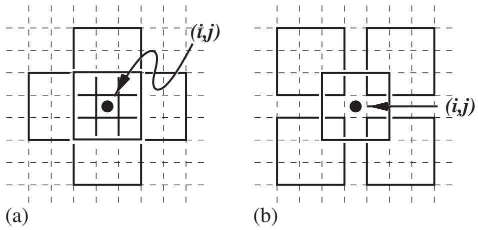


Fig. 4. $Q_{(i,j)}$ and (a) its side-neighbours and (b) its corner-neighbours.

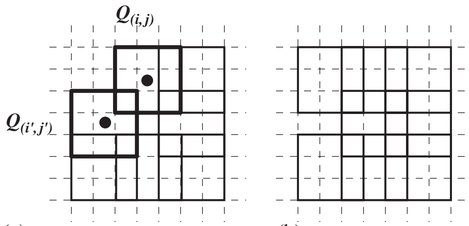


Fig. 5. Side-connected sets of discrete squares. The set (a) is not strongly side connected because $Q_{(i,j)} \cap Q_{(i',j')} \neq \emptyset$ but they are not side-connected and they have no common side-neighbour. (b) A strongly side-connected set.

consecutive elements of the sequence are side-connected. (A sequence consisting of only one square is a side-chain by definition.)

The set Σ is *side-connected* if there is a side-chain in Σ between its any two different elements (see Fig. 5(a)). A maximal side-connected subset of Σ is called a *side-connected component* of Σ. Clearly, the side connected components of Σ give a partition of Σ.

A side-connected set Σ is *strongly side-connected* if whenever $Q_{(i,j)}, Q_{(i',j')} \in \Sigma$ and $Q_{(i,j)} \cap Q_{(i',j')} \neq \emptyset$ then they are side-connected or they have a common side-connected neighbour (see Fig. 5(b)).

Let σ be a set of strongly side-connected components $\Sigma^{(1)}, \dots, \Sigma^{(k)}$ $(k \geqslant 1)$ of the same set Σ. Let $\Sigma^{(l)}, \Sigma^{(l')} \in \sigma$ $(1 \leqslant l, l' \leqslant k)$. $\Sigma^{(l)}$ and $\Sigma^{(l')}$ are *corner-connected* if whenever $Q \in \Sigma^{(l)}$ and $Q' \in \Sigma^{(l')}$ have a common position then Q and Q' are corner-connected squares. (Since $\Sigma^{(l)}$ and $\Sigma^{(l')}$ are maximal, Q and Q' cannot be side-connected squares.) There is a *corner-chain* between $\Sigma^{(l)}$ and $\Sigma^{(l')}$ in σ if a sequence of elements of σ can be selected such that the first element is $\Sigma^{(l)}$ and the last element is $\Sigma^{(l')}$ and any two consecutive elements of the sequence are corner-connected. (A sequence consisting of only one component is a corner-chain by definition.) The set σ is *corner-connected* if there is a corner-chain in σ between its any two elements (see Fig. 6(a)).

4.2.2. Switching patterns

In order to identify not necessarily rectangular parts of binary matrices, we introduce the concept of the *binary pattern* (or shortly, *pattern*) as binary valued function defined on an

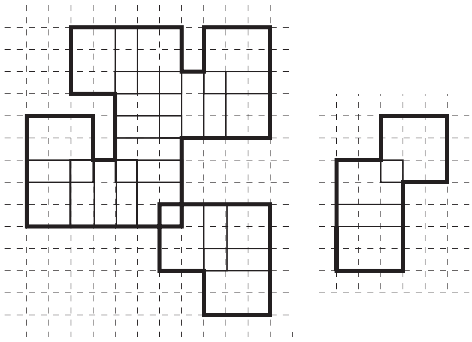


Fig. 6. Two corner-connected sets, both of them have two strongly side-connected components.

arbitrary non-empty subset of $\{1, \ldots, m\} \times \{1, \ldots, n\}$. (In this terminology binary matrices are binary patterns on discrete rectangles.)

Let P be a binary pattern, its domain will be denoted by $\mathrm{dom}(P)$. The absorbed row and column sums of P are denoted by $R_{\beta_0}(P)$ and $S_{\beta_0}(P)$, respectively, where the ith component of $R_{\beta_0}(P)$ is

$$\sum_{(i,j)\in\mathrm{dom}(P)} P(i, j)\beta_0^{-j}$$

for $1 \leqslant i \leqslant m$ and the jth component of $S_{\beta_0}(P)$ is

$$\sum_{(i,j)\in\mathrm{dom}(P)} P(i, j)\beta_0^{-i}$$

for $1 \leqslant j \leqslant n$.

Let us define the binary pattern P' as

$$P'(i, j) = 1 - P(i, j)$$

on $\mathrm{dom}(P)$. By definition, P is a *switching pattern* if

$$R_{\beta_0}(P) = R_{\beta_0}(P') \quad \text{and} \quad S_{\beta_0}(P) = S_{\beta_0}(P').$$

In this case we say that P and P' are a *switching pair*. That is, the patterns of a switching pair can be get from each other by switching their 0s and 1s and still they have the same absorbed row and column sums.

As an example, consider the following binary patterns:

$$E^{(0)}_{(i,j)} = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} \quad \text{and} \quad E^{(1)}_{(i,j)} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{matrix} \tag{29}$$

both of them are defined on the discrete square $Q_{(i,j)}$. It is easy to check that $E^{(0)}_{(i,j)}$ and $E^{(1)}_{(i,j)}$ are a switching pair. They play an important role in the generation of switching patterns, $E^{(0)}_{(i,j)}$ and $E^{(1)}_{(i,j)}$ are called the *0-type* and *1-type elementary switching patterns*, respectively.

4.2.3. Composition of patterns

The *composition* of two patterns P and P' is the function

$$P * P' : \mathrm{dom(P)} \triangle \mathrm{dom(P')} \longrightarrow \{0, 1\}$$

($\triangle$ denotes the symmetric difference) such that

$$[P * P'](i, j) = \begin{cases} P(i, j), & \text{if}\,(i, j) \in \mathrm{dom}(P) \setminus \mathrm{dom}(P'), \\ P'(i, j), & \text{if}\,(i, j) \in \mathrm{dom}(P') \setminus \mathrm{dom}(P). \end{cases}$$

(That is, $P * P'$ is undefined on $\mathrm{dom}(P) \cap \mathrm{dom}(P')$.)

For example,

$$E^{(0)}_{(i,j)} * E^{(0)}_{(i+2,j)} = \begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 0 \\ \text{--} & \text{--} & \text{--} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{array} \tag{30}$$

defined on $Q_{(i,j)} \triangle Q_{(i+2,j)} = \{i-1, i, i+2, i+3\} \times \{j-1, j, j+1\}$. (Just for the sake of simple presentation, on the right side of (30) the whole sub-matrix on the rectangle $\{i-1, \ldots, i+3\} \times \{j-1, j, j+1\}$ is indicated and—denotes the positions in the sub-matrix where the composition is undefined.) Similarly,

$$E^{(0)}_{(i,j)} * E^{(0)}_{(i,j+2)} = \begin{array}{ccccc} 0 & 1 & \text{--} & 1 & 1 \\ 1 & 0 & \text{--} & 0 & 0 \\ 1 & 0 & \text{--} & 0 & 0 \end{array} \tag{31}$$

defined on $Q_{(i,j)} \triangle Q_{(i,j+2)} = \{i-1, i, i+1\} \times \{j-1, j, j+2, j+3\}$. It is easy to see that $E^{(0)}_{(i,j)} * E^{(0)}_{(i+2,j)}$ and $E^{(0)}_{(i,j)} * E^{(0)}_{(i,j+2)}$ are switching patterns, their switching pairs are $E^{(1)}_{(i,j)} * E^{(1)}_{(i+2,j)}$ and $E^{(1)}_{(i,j)} * E^{(1)}_{(i,j+2)}$, respectively, where

$$E^{(1)}_{(i,j)} * E^{(1)}_{(i+2,j)} = \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \text{--} & \text{--} & \text{--} \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{array} \quad \text{and} \quad E^{(1)}_{(i,j)} * E^{(1)}_{(i,j+2)} = \begin{array}{ccccc} 1 & 0 & \text{--} & 0 & 0 \\ 0 & 1 & \text{--} & 1 & 1 \\ 0 & 1 & \text{--} & 1 & 1. \end{array} \tag{32}$$

These examples show how new switching patterns can be created from elementary switching patterns by composition. The following lemma gives a more general way to show how two (not only elementary) switching patterns can be used to generate another switching pattern.

Lemma 2. *Let P_1 and P_2 be switching patterns. If*

$$P_1 = 1 - P_2 \tag{33}$$

on $\mathrm{dom}(P_1) \cap \mathrm{dom}(P_2)$ *then $P_1 * P_2$ is also a switching pattern.*

Proof. Let $I_1 = \mathrm{dom}(P_1)$, $I_2 = \mathrm{dom}(P_2)$, and

$$P = P_1 * P_2$$

on $I = I_1 \triangle I_2$. Let, furthermore,

$$P_1' = 1 - P_1, \quad P_2' = 1 - P_2 \quad \text{and} \quad P' = 1 - P$$

on I_1, I_2, and I, respectively. It is easy to see that

$$P' = P_1' * P_2',$$

on I, and

$$P_1' = 1 - P_2', \tag{34}$$

on $I_1 \cap I_2$.

We are going to show that

$$R_{\beta_0}(P) = R_{\beta_0}(P'). \tag{35}$$

According to the definitions, the ith components of $R_{\beta_0}(P) = (\rho_1, \dots, \rho_m)$, $1 \leqslant i \leqslant m$, can be written as

$$\begin{aligned}
\rho_i &= \sum_{(i,j)\in I} P(i,j)\beta_0^{-j} = \sum_I [P_1 * P_2](i,j)\beta_0^{-j} \\
&= \sum_{I_1 \setminus I_2} P_1(i,j)\beta_0^{-j} + \sum_{I_2 \setminus I_1} P_2(i,j)\beta_0^{-j} \\
&= \left(\sum_{I_1} - \sum_{I_1 \cap I_2} \right) P_1(i,j)\beta_0^{-j} + \left(\sum_{I_2} - \sum_{I_2 \cap I_1} \right) P_2(i,j)\beta_0^{-j}.
\end{aligned}$$

Since P_1 and P_1' are a switching pair on I_1 and P_2 and P_2' are a switching pair on I_2, we get that

$$\begin{aligned}
\rho_i &= \sum_{I_1} P_1'(i,j)\beta_0^{-j} - \sum_{I_1 \cap I_2} P_1(i,j)\beta_0^{-j} + \sum_{I_2} P_2'(i,j)\beta_0^{-j} - \sum_{I_1 \cap I_2} P_2(i,j)\beta_0^{-j} \\
&= \sum_{I_1} P_1'(i,j)\beta_0^{-j} + \sum_{I_2} P_2'(i,j)\beta_0^{-j} - \sum_{I_1 \cap I_2} \big(P_1(i,j) + P_2(i,j)\big)\beta_0^{-j}.
\end{aligned}$$

After partitioning I_1 and I_2 and ordering the new terms we get

$$\begin{aligned}
\rho_i &= \sum_{I_1 \setminus I_2} P_1'(i,j)\beta_0^{-j} + \sum_{I_2 \setminus I_1} P_2'(i,j)\beta_0^{-j} + \sum_{I_1 \cap I_2} \big(P_1'(i,j) + P_2'(i,j)\big)\beta_0^{-j} \\
&\quad - \sum_{I_1 \cap I_2} \big(P_1(i,j) + P_2(i,j)\big)\beta_0^{-j}.
\end{aligned}$$

Then from the definition of the composition (33), and (34) we have

$$\begin{aligned}
\rho_i &= \sum_I [P_1' * P_2'](i,j)\beta_0^{-j} + \sum_{I_1 \cap I_2} 1\beta_0^{-j} - \sum_{I_1 \cap I_2} 1\beta_0^{-j} \\
&= \sum_{(i,j)\in I} P'(i,j)\beta_0^{-j}.
\end{aligned}$$

That is, the ith component of $R_{\beta_0}(P)$ is the same as the ith component of $R_{\beta_0}(P')$. In this way (35) is proved. The equality $S_{\beta_0}(P) = S_{\beta_0}(P')$ can be proved similarly. □

Remark 1. In the case of the compositions of (30) the condition (33) is satisfied, i.e.,

$$E^{(0)}_{(i,j)} = 1 - E^{(0)}_{(i+2,j)}$$

on $Q_{(i,j)} \cap Q_{(i+2,j)}$, therefore $E^{(0)}_{(i,j)} * E^{(0)}_{(i+2,j)}$ is a switching pattern, as we have seen it soon before the proof of Lemma 2. Similar statements are true in the cases of (31) and (32).

4.2.4. Composition of elementary switching patterns

Let $\mathcal{E} = \{E_1, \ldots, E_k\}$ $(k \geqslant 1)$ be a set of elementary switching patterns of the same type on a strongly side-connected set $\Sigma = \{Q_1, \ldots, Q_k\}$. The composition of $\mathcal{E}$ on Σ, denoted by $\mathcal{C}(\mathcal{E})$, is defined as follows. If $k = 1$ then $\mathcal{C}(\mathcal{E}) = E_1$. If $k > 1$ then let us suppose that $Q_1, \ldots, Q_k$ are indexed such that for each $l(< k)$ $\{Q_1, \ldots, Q_l\}$ is a strongly side-connected set and one of its squares is side-connected with Q_{l+1}. (It is easy to see that such an indexing exists.) Then let

$$C = \mathcal{C}(\mathcal{E}) = ((E_1 * E_2) * \cdots) * E_k.$$

Now, we are going to show that this definition is independent from the indexing of $\Sigma = \{Q_1, \ldots, Q_k\}$. There are four cases depending on how many times a position (i, j) is in the sets $\{Q_1, \ldots, Q_k\}$:

(i) If there is exactly one l such that $(i, j) \in Q_l$, then

$$C(i, j) = e^{(l)}_{ij},$$

where $e^{(l)}_{ij}$ denotes the value of E_l in the position (i, j).

(ii) If there are exactly two different l_1 and l_2 such that $(i, j) \in Q_{l_1}$ and $(i, j) \in Q_{l_2}$, then $C(i, j)$ is undefined.

(iii) If there are exactly three different l_1, l_2, and l_3 such that $(i, j) \in Q_{l_1} \cap Q_{l_2} \cap Q_{l_3}$, then two of the elementary switching patterns (say, E_{l_1} and E_{l_3}) has the same value in the position (i, j) and the other one (E_{l_2}) has a different value in the position (i, j), i.e., $e^{(l_1)}_{ij} = e^{(l_3)}_{ij} = 1 - e^{(l_2)}_{ij}$. In this case

$$C(i, j) = e^{(l_1)}_{ij} = e^{(l_3)}_{ij}.$$

(iv) If there are exactly four different l_1, l_2, l_3, and l_4 such that $(i, j) \in Q_{l_1} \cap Q_{l_2} \cap Q_{l_3} \cap Q_{l_4}$, then $C(i, j)$ is undefined.

That is, the value of $C(i, j)$ can be decided simply on the base of the parity of the number of discrete squares of Σ covering (i, j) (independently from the indexing of Σ). Accordingly, if $\mathcal{E}$ is the set of 2D elementary switching patterns of the same type on a strongly side-connected set, then we can simply write $\mathcal{C}(\mathcal{E})$ to denote the compositions of the elements of $\mathcal{E}$. As an example, see Fig. 7.

Lemma 3. *Let $\mathcal{E}$ be a set of elementary switching patterns of the same type on a strongly side-connected set of* 3×3 *squares. Then $\mathcal{C}(\mathcal{E})$ is a switching pattern.*

Proof. The lemma will be proved by induction according to the number of elementary switching patterns in $\mathcal{E}$. Let $E_1, \ldots, E_k$ $(k \geqslant 1)$ denote the elementary switching patterns

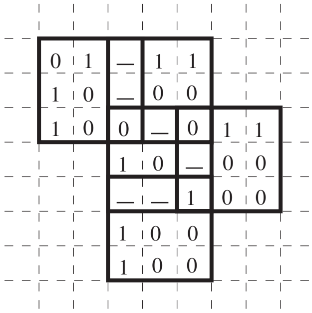


Fig. 7. The composition of the elementary switching patterns $E^{(0)}_{(2,2)}$, $E^{(0)}_{(2,4)}$, $E^{(0)}_{(4,4)}$, $E^{(0)}_{(4,6)}$, and $E^{(0)}_{(6,4)}$. $--$ denotes the positions where the composition is undefined.

of $\mathcal{E}$ on the strongly side-connected set $\Sigma = \{Q_1, \ldots, Q_k\}$. If $k = 1$, i.e., $\mathcal{E} = \{E^{(0)}_{(i,j)}\}$ or $\mathcal{E} = \{E^{(1)}_{(i,j)}\}$ for some (i, j), then Lemma 3 is trivially true. (If $k = 2$ then the possible compositions are just given by (30), (31), and (32), all of them are switching patterns.)

Generally, let us suppose that Lemma 3 is proved for any set $\mathcal{E}$ consisting of k number of elementary switching patterns. Let E_{k+1} be an elementary switching pattern having the same type as $E_1, \ldots, E_k$ such that $\{Q_1, \ldots, Q_k, Q_{k+1}\}$ is also a strongly side-connected set. Let $C = \mathcal{C}(\mathcal{E})$ and $D = C * E_{k+1}$. By induction C is a switching pattern on $\mathrm{dom}(C)$.

In order to apply Lemma 2 we have to show that

$$C(i, j) = 1 - E_{k+1}(i, j) \tag{36}$$

for all $(i, j) \in \mathrm{dom}(C) \cap Q_{k+1}$. There are two cases depending on how many times such a position (i, j) is in the sets $Q_1, \ldots, Q_k$:

(i) There is exactly one $l(\in \{1, \ldots, k\})$ such that $(i, j) \in Q_l$. Then Q_l and Q_{k+1} are side-connected. Since E_l and E_{k+1} are elementary switching patterns of the same type, it is easy to check that (36) is satisfied if Q_{k+1} is either north, east, south, or west side-connected to Q_l (see Fig. 8).

(ii) There are exactly three l_1, l_2, and l_3, all of them from $\{1, \ldots, k\}$, such that $(i, j) \in Q_{l_1} \cap Q_{l_2} \cap Q_{l_3}$. Then one of them, say, Q_{l_2} is side-connected to the other two squares and Q_{l_1} and Q_{l_3} are corner-connected squares. Since E_{l_1}, E_{l_2}, E_{l_3}, and E_{k+1} are elementary switching patterns of the same type, it is easy to check that (36) is satisfied if Q_{k+1} is either north-east, south-east, south-west, or north-west corner-connected to Q_l (see Fig. 9).

(If (i, j) is in even number of squares of $Q_1, \ldots, Q_k$ then $(i, j) \notin \mathrm{dom}(C) \cap Q_{k+1}$, see the cases (ii) and (iv) in the paragraph introducing the definition of the composition of elementary switching patterns.)

That is, (36) is proved for every possible cases. Then Lemma 2 can be applied, therefore, D is a switching pattern. □

Henceforth, the switching patterns constructed from elementary switching patterns of the same type by composition are called *composite switching patterns*. We say that the

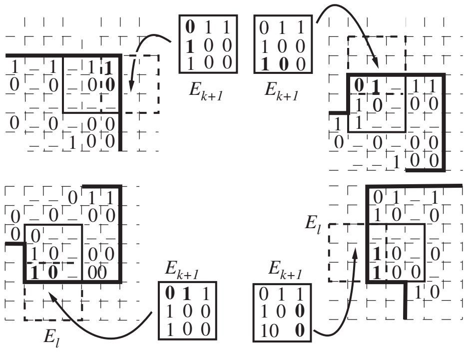


Fig. 8. Illustration of the fact that Eq. (36) is satisfied if exactly one of the elementary switching patterns of C, say E_l, has common positions with the elementary switching pattern E_{k+1} (the binary values at the common positions are indicated by bold digits) and E_l and E_{k+1} have the same type (here: 0-type; 1-type patterns can be checked similarly). The positions indicated by "_ _" do not belong to the domain of C.

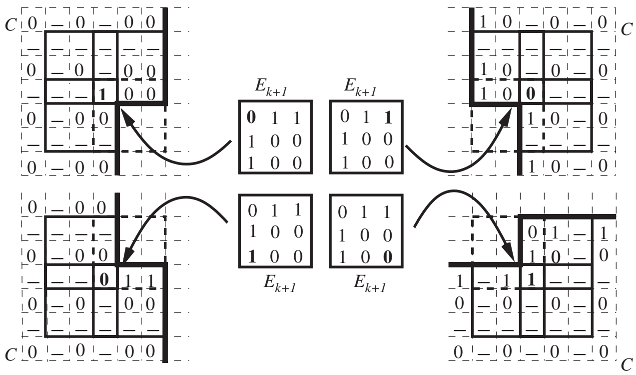


Fig. 9. Illustration of the fact that Eq. (36) is satisfied if exactly three of the elementary switching patterns of C have common positions with the elementary switching pattern E_{k+1} (the binary value at the common position is indicated by bold digit) and all three elementary switching patterns of C and E_{k+1} have the same type (here: 0-type; 1-type patterns can be checked similarly). The positions indicated by "_ _" do not belong to the domain of C.

composite switching pattern has 0-type/1-type if it is the composition of 0-type/1-type elementary switching patterns, respectively. For example, (30)–(32) and Fig. 7 show composite switching patterns of types 0, 0, 1, 1, and 0, respectively.

4.2.5. Composition of corner-connected components

Lemma 4. *Let $\Sigma^{(0)}$ and $\Sigma^{(1)}$ be corner-connected components. Let $C^{(0)}$ and $C^{(1)}$ be* 0-*type and* 1-*type* 2*D composite switching patterns on $\Sigma^{(0)}$ and on $\Sigma^{(1)}$, respectively. Then $C^{(0)} * C^{(1)}$ is a switching pattern.*

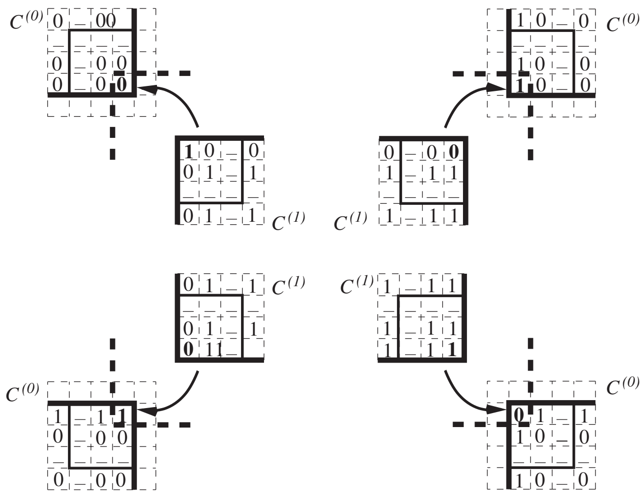


Fig. 10. Illustration of the fact that Eq. (37) is satisfied in the common position of corner-connected components. The domains of E' and E'' are indicated in $C^{(0)}$ and $C^{(1)}$, respectively, by bold squares with solid lines.

Proof. In order to use Lemma 2 we have to show that

$$C^{(0)} = 1 - C^{(1)} \tag{37}$$

on $\mathrm{dom}(C^{(0)}) \cap \mathrm{dom}(C^{(1)})$. Since $\Sigma^{(0)}$ and $\Sigma^{(1)}$ are corner-connected components, $\mathrm{dom}(C^{(0)}) \cap \mathrm{dom}(C^{(1)})$ consists of the common positions of corner-connected squares of $\Sigma^{(0)}$ and $\Sigma^{(1)}$. Formally, if $(i, j) \in \mathrm{dom}(C^{(0)}) \cap \mathrm{dom}(C^{(1)})$ then there are 2D elementary switching patterns E' and E'' such that $(i, j) \in \mathrm{dom}(E') \cap \mathrm{dom}(E'')$. It is easy to check that for such a position (i, j)

$$E'(i, j) = 1 - E''(i, j)$$

in all possible cases (see Fig. 10). That is, (37) is satisfied. Then applying Lemma 2 to the 2D composite switching patterns $C^{(0)}$ and $C^{(1)}$ we get that $C^{(0)} * C^{(1)}$ is a switching pattern. $\square$

Let σ be a corner-connected set of the strongly side-connected components $\Sigma^{(1)}, \dots, \Sigma^{(k)}$ $(k \geqslant 1)$. Let $C^{(1)}, \dots, C^{(k)}$ be 0- or 1-type composite switching patterns on $\Sigma^{(l)}, \dots, \Sigma^{(k)}$, respectively. Let us suppose also that if $\Sigma^{(l)}$ and $\Sigma^{(l')}$ are corner-connected then $C^{(l)}$ and $C^{(l')}$ have different type. In this case we say that $\gamma = \{C^{(1)}, \dots, C^{(k)}\}$ is a set of *alternatively corner-connected components* on σ.

The composition of γ on σ, denoted by $\mathcal{C}(\gamma)$, is defined as follows. If $k = 1$ then $\mathcal{C}(\gamma) = C^{(1)}$. If $k > 1$ then let us suppose that $\Sigma^{(1)}, \dots, \Sigma^{(k)}$ are indexed such that for each $l\,(l < k)$ $\{\Sigma^{(1)}, \dots, \Sigma^{(k)}\}$ is a corner-connected set and one of its elements is corner-connected with $\Sigma^{(l+1)}$. (It is easy to see that such an indexing exists.) Then let

$$C = \mathcal{C}(\gamma) = ((C^{(1)} * C^{(2)}) * \cdots) * C^{(k)}.$$

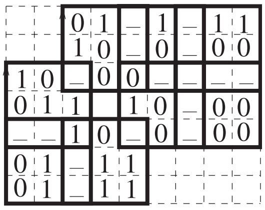


Fig. 11. Composition of two alternatively corner-connected components.

It is easy to show that this definition is independent from the indexing of σ, because $\mathrm{dom}(C) = \cup_l \Sigma^{(l)} \setminus \cup_{l,l'}(\Sigma^{(l)} \cap \Sigma^{(l')})$ and $C(i, j) = C^{(l)}(i, j)$, where $l \in \{1, \ldots, k\}$ is the only index such that $(i, j) \in \Sigma^{(l)}$. Henceforth, we may simply write $\mathcal{C}(\gamma)$ to denote the composition of the elements of γ. As an example, see Fig. 11.

Theorem 1. *Let* $\gamma = \{C^{(1)}, \ldots, C^{(k)}\}$ $(k \geqslant 1)$ *be a set of alternatively corner-connected components. Then* $\mathcal{C}(\gamma)$ *is a composite switching pattern.*

Proof. It follows from Lemma 4 directly. $\square$

5. Discussion

A new kind of discrete tomography problem was introduced, the reconstruction of discrete sets from their absorbed projections. A special case of this problem was discussed in this paper, namely, the uniqueness of 2D binary matrices with respect to their absorbed row and column sums when the absorption is represented by the constant β_0 of (7). It was shown that the uniqueness in this case could be characterised similarly as in the classical case of reconstruction without absorption, because similar elementary switchings could be given by $E^{(0)}_{(i,j)}$ and $E^{(1)}_{(i,j)}$.

We have shown that there is a hierarchy of 2D switching patterns in the case of absorption. The most simple kind of switching patterns are the 0-type and 1-type elementary switching patterns given by (29). More complicated switching patterns, so-called 0- and 1-type composite switching patterns can be constructed from 0-type and 1-type elementary switching patterns, respectively, defined on strongly side-connected components of 3×3 discrete squares by composition (Lemma 3). Finally, we get the most complex composite switching patterns as the composition of alternatively corner-connected components (Lemma 4).

Theorem 1 also means that if a binary matrix contains an alternatively corner-connected component then it is non-uniquely determined with respect to its absorbed row and column sums. That is, the existence of an alternatively corner-connected component is sufficient for the non-uniqueness in the case of problem UNIQUENESS $DA2D(\beta_0, A)$. In another paper [10] we could prove that this condition is also necessary for the non-uniqueness, that is, since this condition is necessary and sufficient for the non-uniqueness, we have also that the compositions of alternatively corner-connected components are the most general switching patterns.

Acknowledgements

A part of this research was done while the first author visited the Laboratoire d'Informatique Algorithmique: Fondements et Applications, Université Paris 7 Denis-Diderot, Paris, France. Special thanks to Prof. Gabor T. Herman (Graduate College, City University of New York) and Laurent Vuillon (LIAFA, Université Paris 7) for their valuable comments and discussions. This work was supported by the Grant OTKA T 032241 and NSF Grant DMS 0306215 (Aspects of Discrete Tomography).

References

[2] E. Balogh, A. Kuba, A. Del Lungo, M. Nivat, Reconstruction of binary matrices from absorbed projections, in: A. Braquelaire, J.-O. Lachaud, A. Vialard (Eds.), Discrete Geometry for Computer Imagery, Lecture Notes in Computer Science, Vol. 2301, Springer, Berlin, 2002, pp. 392–403.

[3] E. Barcucci, A. Frosini, S. Rinaldi, Reconstruction of discrete sets from two absorbed projections: an algorithm, Electron. Notes Discrete Math. 12 (2003) 12.

[4] R.A. Brualdi, Matrices of zeros and ones with fixed row and column sums, Linear Algebra Appl. 33 (1980) 159–231.

[6] G.T. Herman, A. Kuba (Eds.), Discrete Tomography: Foundations, Algorithms and Applications, Birkhäuser, Boston, 1999.

[7] G.T. Herman, R. Marabini, J.-M. Carazo, E. Garduno, R.M. Lewitt, S. Matej, Image processing approaches to biological three-dimensional electron microscopy, Int. J. Imaging Syst. Technol. 11 (2000) 12–29.

[8] A. Kuba, G.T. Herman, Discrete tomography: A historical overview, in: Discrete Tomography: Foundations Algorithms and Applications, Birkhäuser, Boston, 1999, pp. 3–34.

[9] A. Kuba, G.T. Herman, S. Matej, A. Todd-Pokropek, Medical applications of discrete tomography, in: D.Z. Du, P.M. Pardalos, J. Wang (Eds.), Discrete Mathematical Problems with Medical Applications, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol. 55, AMS, Rhode Island, 2000, pp. 195–208.

[10] A. Kuba, M. Nivat, Reconstruction of discrete sets with absorption, Linear Algebra Appl. 339 (2001) 171–194.

[11] M. Lothaire, Combinatorics on Words, Cambridge University Press, Cambridge, 1997.

[12] H.J. Ryser, Combinatorial properties of matrices of zeros and ones, Canad. J. Math. 9 (1957) 371–377.

[13] P. Schwander, C. Kisielowski, M. Seibt, F.H. Baumann, Y. Kim, A. Ourmazd, Mapping projected potential, interfacial roughness, and composition in general crystalline solids by quantitative transmission electron microscopy, Phys. Rev. Lett. 71 (1993) 4150–4153.